CLASS –XII MATHEMATICS NCERT SOLUTIONS
Determinants
Exercise 4.2

Answers

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2(-1)-4(-5) = -2+20 = 18$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta.\cos\theta-(-\sin\theta).\sin\theta = \cos^2\theta+\sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix} = (x^2-x+1)(x+1)-(x+1)(x-1)$

$= (x^3+1)-(x^2-1) = x^3-x^2+2$

3. Given: A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then 2A = $2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2\times1 & 2\times2 \\ 2\times4 & 2\times2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

L.H.S. = $|2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2\times4-4\times8 = 8-32 = -24$

R.H.S. = $4|A| = 4\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 4(1\times2-4\times8) = 4(-6) = -24$

Since L.H.S. = R.H.S.
Hence, proved.

4. Given: A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then 3A = $3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$

L.H.S. = $|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix} = 3(36-0)$ = 3 x 36 = 108

R.H.S. = $27|A| = 27\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$ = 27 [1 (4 – 0)] = 27 x 4 = 108

Since L.H.S. = R.H.S.
Hence, proved.

5. Evaluate the determinants:

(i) Given: $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Expanding along first row, $3\begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$

$= 3(0-5)+1\{0-(-3)\}-2(0-0)$

$= -15+3-0 = -12$

(ii) Given: $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

Expanding along first row, $3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

$= 3(1+6)+4\{1-(-4)\}+5(3-2)$

$= 3\times7+4\times5+5\times1$

= 21 + 20 + 5 = 46

(iii) Given: $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$

Expanding along first row, $0\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ -2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix}$

$= 0(0+9)-(0-6)+2(-3-0)$

= 0 + 6 – 6 = 0

(iv) Given: $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

Expanding along first row, $2\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix}$

$= 2(0-5)+(0+3)-2(0-6)$

$= -10+3+12 = 5$

6. Given: $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, $\Rightarrow$ $|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{vmatrix}$

Expanding along first row, $1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$

$= \{-9-(-12)\}-\{-18-(-15)\}-2(8-5)$

$= -9+12-(-18+15)-2(3)$

$= 3-(-3)-6$

$= 3+3-6 = 0$

7. (i) Given: $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow \quad 2-20 = 2x^2 - 24$

$\Rightarrow \quad -18 = 2x^2 - 24$

$\Rightarrow \quad 2x^2 = -18 + 24$

$\Rightarrow \quad 2x^2 = 6$

$\Rightarrow \quad x^2 = 3$

$\Rightarrow \quad x = \pm\sqrt{3}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow \quad 10-12 = 5x - 6x$

$\Rightarrow \quad -2 = -x$

$\Rightarrow \quad x = 2$

8. Given: $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

$\Rightarrow \quad x^2 - 36 = 36 - 36$

$\Rightarrow \quad x^2 - 36 = 0$

$\Rightarrow \quad x^2 = 36$

$\Rightarrow \quad x = \pm 6$

Therefore, option (B) is correct.

Determinants

Exercise 4.2

Answers

1. Given: $\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$

Operating $C_1 \to C_1 + C_2$ $\begin{vmatrix} x+a & a & x+a \\ y+b & b & y+b \\ z+c & c & z+c \end{vmatrix} = 0 \Rightarrow$ 0 = 0 [$\because$ C_1 and C_2 are identical]

$\Rightarrow$ L.H.S. = R.H.S.

2. On $\begin{vmatrix} a-b & b-c & a-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ Operating $C_1 \to C_1 + C_2 + C_3$

= $\begin{vmatrix} a-b+b-c+c-a & b-c & a-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$ = $\begin{vmatrix} 0 & b-c & a-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$ = 0 = R.H.S.

[$\because$ All entries of one column here first are zero]

3. On $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$, operating $C_3 \to C_3 - C_1$

= $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix}$ = $9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$ = 9 x 0 = 0 [$\because$ two columns are identical] Proved.

4. Given: $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ = $\begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ba \\ 1 & ab & ca+cb \end{vmatrix}$

Operating $C_3 \to C_3 + C_2$ $\begin{vmatrix} 1 & bc & ab+ab+ac \\ 1 & ca & ab+ab+ac \\ 1 & ab & ab+ab+ac \end{vmatrix}$ = $(ab+ab+ac)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$

= $(ab+ab+ac)(0)$ = 0 [$\because$ two columns are identical] Proved.

5. L.H.S. = $\begin{vmatrix} (b+c) & q+r & y+z \\ (c+a) & r+p & z+x \\ (a+b) & p+q & x+y \end{vmatrix}$ operating $R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} b+c+c+a+a+b & q+r+r+p+p+q & y+z+z+x+x+y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

$= \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$

$= 2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$ [operating $R_1 \to R_1 - R_2$]

$= 2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$ [operating $R_3 \to R_3 - R_1$]

$= 2\begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$ [operating $R_2 \to R_2 - R_3$]

$= -2\begin{vmatrix} b & q & y \\ a & p & x \\ c & r & z \end{vmatrix}$ [Interchanging R_2 and R_3]

$= -(-2)\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ [Interchanging R_2 and R_3] $= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$ = R.H.S.

6. Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = \Delta = (-1)^3 \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix}$ [Taking (-1) common from each row]

Interchanging rows and columns in the determinants on R.H.S.,

$\Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \quad \Rightarrow \quad \Delta = -\Delta \quad \Rightarrow \quad \Delta + \Delta = 0$

$\Rightarrow \quad 2\Delta = 0 \quad \Rightarrow \quad \Delta = 0$ Proved.

7. L.H.S. = $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

Taking common a,b,c from R_1, R_2, R_3 respectively,

$= abc\begin{vmatrix} -a & a & a \\ a & -b & b \\ a & b & -c \end{vmatrix} = abc\begin{vmatrix} 0 & 0 & 2c \\ a & -b & c \\ a & b & -c \end{vmatrix}$ [operating $R_1 \to R_1 + R_2$]

$= abc.2c\begin{vmatrix} a & -b \\ a & b \end{vmatrix} = abc.2c(ab+ab) = abc.2c.2ab = 4a^2b^2c^2$ = R.H.S.

8. (i) L.H.S. = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c-a & c^2-a^2 \end{vmatrix} = 1\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$ [Expanding along 1st column]

$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a)$ = R.H.S. Proved.

(ii) L.H.S. = $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$ operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} = 1\begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$

$= 1\begin{vmatrix} b-a & c-a \\ (b-a)(b^2+a^2+ab) & (c-a)(c^2+a^2+ac) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ (b^2+a^2+ab) & (c^2+a^2+ac) \end{vmatrix}$

$= (b-a)(c-a)(c^2+a^2+ac-b^2-a^2-ab) = (b-a)(c-a)(c^2-b^2+ac-ab)$

$= (b-a)(c-a)[(c-b)(c+b)a(c-b)] = (b-a)(c-a)(c-b)(c+b+a)$

$= -(a-b)(c-a)[-(c-b)(c+b+a)] = (a-b)(b-c)(c-a)(a+b+c)$ = R.H.S.

9. $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

L.H.S. = $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$ = $\begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$ [Multiplying R_1, R_2,R_3 by x, y, z respectively]

= $\dfrac{xyz}{xyz}\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$ = $\begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$

= $\begin{vmatrix} x^2 & x^3 & 1 \\ y^2-x^2 & y^3-x^3 & 0 \\ z^2-x^2 & z^3-x^3 & 0 \end{vmatrix}$ [operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

= $1\begin{vmatrix} y^2-x^2 & y^3-x^3 \\ z^2-x^2 & z^3-x^3 \end{vmatrix}$ = $\begin{vmatrix} (y-x)(y+x) & (y-x)(y^2+x^2+yx) \\ (z-x)(z+x) & (z-x)(z^2+x^2+zx) \end{vmatrix}$

= $(y-x)(z-x)\begin{vmatrix} y+x & y^2+x^2+yx \\ z+x & z^2+x^2+zx \end{vmatrix}$

= $(y-x)(z-x)\left[(y+x)(z^2+x^2+zx)-(z+x)(y^2+x^2+xy)\right]$

= $(y-x)(z-x)\left[yz^2+yx^2+xyz+xz^2+x^3+x^2z-zy^2-zx^2-xyz-xy^2-x^3-x^2y\right]$

= $(y-x)(z-x)\left[yz^2-zy^2+xz^2-xy^2\right]$ = $(y-x)(z-x)\left[yz(z-y)+x(z^2-y^2)\right]$

= $(y-x)(z-x)\left[yz(z-y)+x(z-y)(z+y)\right]$ = $(y-x)(z-x)(z-y)\left[yz+x(z+y)\right]$

= $-(x-y)(z-x)\left[-(y-z)\right]\left[yz+xz+xy\right]$ = $(x-y)(y-z)(z-x)(xy+yz+zx)$ = R.H.S.

10. (i) L.H.S. = $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ = $\begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$ $[R_1 \to R_1 + R_2 + R_3]$

= $(5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

= $(5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix}$ [operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$= (5x+4).1\begin{vmatrix}4-x & 0\\ 0 & 4-x\end{vmatrix} = (5x+4)(4-x)^2 = \text{R.H.S.}$$

(ii) L.H.S. $= \begin{vmatrix} y+k & y & y\\ y & y+k & y\\ y & 2x & y+k\end{vmatrix} = \begin{vmatrix} 3y+k & y & y\\ 3y+k & y+k & y\\ 3y+k & y & y+k\end{vmatrix}$ $[C_1 \to C_1 + C_2 + C_3]$

$$= (3y+k)\begin{vmatrix} 1 & y & y\\ 1 & y+k & y\\ 1 & y & y+k\end{vmatrix}$$

$= (3y+k)\begin{vmatrix} 1 & y & y\\ 0 & k & 0\\ 0 & 0 & k\end{vmatrix}$ [operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$= (3y+k).1\begin{vmatrix} k & 0\\ 0 & k\end{vmatrix} = (3y+k)k^2 = k^2(3y+k) = \text{R.H.S.}$ Proved.

11. (i) L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$

$= \begin{vmatrix} a+b+c & a+b+c & a+b+c\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$ $[R_1 \to R_1 + R_2 + R_3]$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

$= (a+b+c)\begin{vmatrix} 1 & 0 & 0\\ 2b & -b-c-a & 0\\ 2c & 0 & -c-a-b\end{vmatrix}$ $[C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1]$

$$= (a+b+c).1.\begin{vmatrix} -b-c-a & 0\\ 0 & -c-a-b\end{vmatrix} = (a+b+c)\{-(b+c+a)\}\{-(c+a+b)\}$$

$= (a+b+c)^3 = \text{R.H.S.}$ Proved.

(ii) L.H.S. $= \begin{vmatrix} x+y+2z & x & y\\ z & y+z+2x & y\\ z & x & z+x+2y\end{vmatrix}$

$= \begin{vmatrix} 2(x+y+z) & x & y\\ 2(x+y+z) & y+z+2x & y\\ 2(x+y+z) & x & z+x+2y\end{vmatrix}$ $[C_1 \to C_1 + C_2 + C_3]$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$= 2(x+y+z).1.\begin{vmatrix} x+y+z & 0 \\ 0 & x+y+z \end{vmatrix} = 2(x+y+z)\left[(x+y+z)^2 - 0\right]$$

$$= 2(x+y+z)^3 = \text{R.H.S.}$$ Proved.

12. L.H.S. = $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \qquad [R_1 \to R_1 + R_2 + R_3]$

$$= (1+x+x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1+x+x^2)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix} \quad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= (1+x+x^2).1.\begin{vmatrix} 1-x^2 & x-x^2 \\ x^2-x & 1-x \end{vmatrix} = (1+x+x^2).1.\begin{vmatrix} (1-x)(1+x) & x(1-x) \\ -x(1-x) & 1-x \end{vmatrix}$$

$$= (1+x+x^2)\left[(1-x)^2(1+x) + x^2(1-x)^2\right] = (1+x+x^2)(1-x)^2(1+x+x^2)$$

$$= (1+x+x^2)^2(1-x)^2 = \left[(1+x+x^2)(1-x)\right]^2 = (1-x+x-x^2+x^2-x^3)^2$$

$$= (1-x^3)^2 = \text{R.H.S.}$$ Proved.

13. L.H.S. = $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} \qquad [C_1 \to C_1 - b\,C_3 \text{ and } C_2 \to C_2 + a\,C_3]$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix} \quad [R_3 \to R_3 - b\,R_1]$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix} = (1+a^2+b^2)^2(1-a^2+b^2+2a^2) = (1+a^2+b^2)^3 = \text{R.H.S.}$$

14. L.H.S. = $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$

Multiplying C_1, C_2, C_3 by a, b, c respectively and then dividing the determinant by abc,

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad [R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$= (1+a^2+b^2+c^2)(1)(1-0) = 1+a^2+b^2+c^2$ = R.H.S. Proved.

15. Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 x 3. ....(i)

$$\therefore \quad kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} \quad \Rightarrow \quad |kA| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

$$\Rightarrow \quad |kA| = k^3\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = k^3|A| \qquad \text{[From eq. (i)]}$$

Therefore, option (C) is correct.

16. Since, Determinant is a number associated to a square matrix.
Therefore, option (C) is correct.

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Determinants
Exercise 4.3

Answers

1. (i) Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ = $\left|\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{bmatrix}\right|$

$= \left|\frac{1}{2}\left[1(0-3)-0(6-4)+1(18-0)\right]\right| = \left|\frac{1}{2}(-3+18)\right| = \left|\frac{15}{2}\right| = \frac{15}{2}$ sq. units

(ii) Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ = $\left|\frac{1}{2}\begin{bmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{bmatrix}\right|$

$= \left|\frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]\right| = \left|\frac{1}{2}\left[2(-7)-7(-9)-2\right]\right|$

$= \left|\frac{1}{2}(-14+63-2)\right| = \left|\frac{1}{2}(63-16)\right| = \left|\frac{47}{2}\right| = \frac{47}{2}$ sq. units

(iii) Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ = $\left|\frac{1}{2}\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}\right|$

$= \left|\frac{1}{2}\left[-2(2+8)-(-3)(3+1)+1(-24+2)\right]\right| = \left|\frac{1}{2}\left[-2(10)+3(4)-22\right]\right|$

$= \left|\frac{1}{2}(-20+12-22)\right| = \left|\frac{1}{2}\times(-30)\right| = \left|\frac{1}{2}\times 30\right| = 15$ sq. units

2. Area of triangle ABC = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ = $\left|\frac{1}{2}\begin{bmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{bmatrix}\right|$

$= \left|\frac{1}{2}\left[a(c+a-a-b)-(b+c)(b-c)+1\{b(a+b)-c(c+a)\}\right]\right|$

$= \left|\frac{1}{2}\left[a(c-b)-(b^2-c^2)+(ab+b^2-c^2-ac)\right]\right| = \left|\frac{1}{2}(ac-ab-b^2+c^2+ab+b^2-c^2-ac)\right|$

$= \left|\frac{1}{2}\times 0\right| = 0$

Therefore, points A, B and C are collinear.

3. (i) Given: Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$

$\Rightarrow$ Modulus of $\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$

$\Rightarrow \left|\frac{1}{2}\left[k(0-2)-0+1(8-0)\right]\right| = 4 \qquad \Rightarrow \left|\frac{1}{2}(-2k+8)\right| = 4$

$\Rightarrow |-k+4| = 4 \qquad \Rightarrow -k+4 = \pm 4$

Taking positive sign, $-k+4=4 \qquad \Rightarrow k=0$

Taking negative sign, $-k+4=-4 \qquad \Rightarrow k=8$

(ii) Given: Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 4$

$\Rightarrow \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4 \qquad \Rightarrow \left|\frac{1}{2}\left[-2(4-k)-0+1(0-0)\right]\right| = 4$

$\Rightarrow \left|\frac{1}{2}(-8+2k)\right| = 4 \qquad \Rightarrow |-k+4| = 4 \qquad \Rightarrow -k+4 = \pm 4$

Taking positive sign, $-k+4=4 \qquad \Rightarrow k=0$

Taking negative sign, $-k+4=-4 \qquad \Rightarrow k=8$

4. (i) Let $P(x, y)$ be any point on the line joining the points (1, 2) and (3, 6).

Then, Area of triangle that could be formed by these points is zero.

$\therefore$ Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ Modulus of $\frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \qquad \Rightarrow \frac{1}{2}\left[x(2-6)-y(1-3)+1(6-6)\right]=0$

$\Rightarrow -4x+2y=0 \qquad \Rightarrow -2x+y=0$

$\Rightarrow y=2x$ which is required line.

(ii) Let $P(x, y)$ be any point on the line joining the points (3, 1) and (9, 3).

Then, Area of triangle that could be formed by these points is zero.

$\therefore$ Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ Modulus of $\frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$ $\Rightarrow$ $\frac{1}{2}\left[x(1-3)-y(3-9)+1(9-9)\right]=0$

$\Rightarrow$ $-2x+6y=0$ $\Rightarrow$ $-x+3y=0$

$\Rightarrow$ $x-3y=0$ which is required line.

5. Given: Area of triangle = Modulus of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 35$

$\Rightarrow$ Modulus of $\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = 35$ $\Rightarrow$ $\left|\frac{1}{2}\left[2(4-4)-(-6)(5-k)+1(20-4k)\right]\right|=35$

$\Rightarrow$ $\left|\frac{1}{2}[0+30-6k+20-4k]\right|=35$ $\Rightarrow$ $\left|\frac{1}{2}[50-10k]\right|=35$

$\Rightarrow$ $|25-5k|=35$ $\Rightarrow$ $25-5k=\pm 35$

Taking positive sign, $25-5k=35$ $\Rightarrow$ $k=-2$

Taking negative sign, $25-5k=-35$ $\Rightarrow$ $k=12$

Therefore, option (D) is correct.

Determinants

Exercise 4.4

Answers

1. (i) Let $\Delta = \begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

M_{11} = Minor of $a_{11} = |3| = 3$ and $A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

M_{12} = Minor of $a_{12} = |0| = 0$ and $A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

M_{21} = Minor of $a_{21} = |-4| = 4$ and $A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

M_{22} = Minor of $a_{22} = |2| = 2$ and $A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) Let $\Delta = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

M_{11} = Minor of $a_{11} = |d| = d$ and $A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

M_{12} = Minor of $a_{12} = |b| = b$ and $A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$

M_{21} = Minor of $a_{21} = |c| = c$ and $A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

M_{22} = Minor of $a_{22} = |a| = a$ and $A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

2. (i) Let $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

M_{11} = Minor of $a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and $A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (1) = 1$

M_{12} = Minor of $a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$ and $A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

M_{13} = Minor of $a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ and $A_{13} = (-1)^{1+3} M_{13} = (-1)^4 (0) = 0$

M_{21} = Minor of $a_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$ and $A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (0) = 0$

M_{22} = Minor of $a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and $A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$

M_{23} = Minor of $a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ and $A_{23} = (-1)^{2+3} M_{23} = (-1)^5 (0) = 0$

M31 = Minor of $a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$ and A31 = $(-1)^{3+1} M_{31} = (-1)^4 (0) = 0$

M32 = Minor of $a_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$ and A32 = $(-1)^{3+2} M_{32} = (-1)^5 (0) = 0$

M33 = Minor of $a_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and A31 = $(-1)^{3+3} M_{33} = (-1)^6 (1) = 1$

(ii) Let $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -5 \\ 0 & 2 & 2 \end{vmatrix}$

M11 = Minor of $a_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 - (-1) = 11$ and A11 = $(-1)^{1+1} M_{11} = (-1)^2 (11) = 11$

M12 = Minor of $a_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6$ and A12 = $(-1)^{1+2} M_{12} = (-1)^3 (6) = -6$

M13 = Minor of $a_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$ and A13 = $(-1)^{1+3} M_{13} = (-1)^4 (3) = 3$

M21 = Minor of $a_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0 - 4 = -4$ and A21 = $(-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

M22 = Minor of $a_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$ and A22 = $(-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

M23 = Minor of $a_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$ and A23 = $(-1)^{2+3} M_{23} = (-1)^5 (1) = -1$

M31 = Minor of $a_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0 - 20 = -20$ and A31 = $(-1)^{3+1} M_{31} = (-1)^4 (-20) = -20$

M32 = Minor of $a_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1 - 12 = -13$ and A32 = $(-1)^{3+2} M_{32} = (-1)^5 (-13) = 13$

M33 = Minor of $a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$ and A31 = $(-1)^{3+3} M_{33} = (-1)^6 (5) = 5$

3. Elements of second row of Δ are $a_{21} = 2, a_{22} = 0, a_{23} = 1$

A21 = Cofactor of $a_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix}$ = $(-1)^3 (9 - 16) = -(-7) = 7$

A22 = Cofactor of $a_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix}$ = $(-1)^4 (15 - 8) = 7$

A23 = Cofactor of $a_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$ = $(-1)^5(10-3) = -7$

$\therefore$ $\Delta = a_{21} + A_{21} + a_{22}A_{22} + a_{23}A_{23}$ = $2(7)+0(7)+1(-7) = 14-7 = 7$

4. Elements of third column of Δ are $a_{13} = yz, a_{23} = zx, a_{33} = xy$

A13 = Cofactor of $a_{13} = (-1)^{1+3}\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix}$ = $(-1)^4(z-y) = z-y$

A23 = Cofactor of $a_{23} = (-1)^{2+3}\begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix}$ = $(-1)^5(z-x) = x-z$

A33 = Cofactor of $a_{33} = (-1)^{3+3}\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}$ = $(-1)^6(y-x) = y-x$

$\therefore$ $\Delta = a_{13} + A_{13} + a_{23}A_{23} + a_{33}A_{33}$ = $yz(z-y)+zx(x-z)+xy(y-x)$

= $yz^2 - y^2z + zx^2 - xz^2 + xy^2 - x^2y$ = $(yz^2 - y^2z) + (xy^2 - xz^2) + (xz^2 - x^2y)$

= $yz(z-y) + x(y^2 - z^2) - x^2(y-z)$ = $(y-z)\left[-yz + x(y+z) - x^2\right]$

= $(y-z)(-yz + xy + xz - x^2)$ = $(y-z)\left[-y(z-x) + x(z-x)\right]$

= $(y-z)(z-x)(-y+x)$ = $(x-y)(y-x)(z-x)$

5. Option (D) is correct.

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Exercise 4.5

Answers

1. Here A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\Rightarrow$ $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$\therefore$ A_{11} = Cofactor of $a_{11} = (-1)^2(4) = 4$

A_{12} = Cofactor of $a_{12} = (-1)^3(3) = -3$

A_{21} = Cofactor of $a_{21} = (-1)^3(2) = -2$

A_{22} = Cofactor of $a_{22} = (-1)^4(1) = 1$

$\therefore$ adj. A = $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ = $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

2. Here A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$ $\Rightarrow$ $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix}$

$\therefore$ $A_{11} = +\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3$ $\quad$ $A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}$ $= -(2+10) = -12$

$A_{13} = +\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$ $\quad$ $A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - (-1) = 1$

$A_{22} = +\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$ $\quad$ $A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(-2) = 2$

$A_{31} = +\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11$ $\quad$ $A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$

$A_{33} = +\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$

$\therefore$ adj. A = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & -12 & 5 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

3. Let A = $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ $\Rightarrow$ adj. A = $\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow$ A.(adj. A) = $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ = $\begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix}$ = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .....(i)

Again (adj. A). A = $\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} = \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .....(ii)

And $|A| = \begin{vmatrix} 2 & 3 \\ -4 & -6 \end{vmatrix} = 2(-6)-3(-4) = -12+12 = 0$

Again $|A|I = |A|I_2 = (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .....(iii)

$\therefore$ From eq. (i), (ii) and (iii) A. (adj. A) = (adj. A). A = $|A|I$

4. Let A = $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$ $\Rightarrow$ $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$

$\therefore$ $A_{11} = +\begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = +0+0 = 0$ $A_{12} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -(9+2) = -11$

$A_{13} = +\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = +(0-0) = 0$ $A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} - (-3-0) = 3$

$A_{22} = +\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3-2 = 1$ $A_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0+1) = -1$

$A_{31} = +\begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2-0 = 2$ $A_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2-6) = 8$

$A_{33} = +\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3+0 = 3$

$\therefore$ adj. A = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$

$\therefore$ A. (adj. A) = $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0-0-0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$

= $\begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$ ....(i)

Again (adj. A). A = $\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0-0+0 & 0+2+9 \end{bmatrix}$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \qquad \text{..........(ii)}$$

And $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 1(0-0)-(-1)(9+2)+2(0-0) = 0+11+0 = 11$

Also $|A|I = |A|I_3 = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

..........(iii)

$\therefore$ From eq. (i), (ii) and (iii) A. (adj. A) = (adj. A). A = $|A|I$

5. Let A = $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ $\therefore$ $|A| = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} = 6-(-8) = 6+8 = 14 \neq 0$

$\therefore$ Matrix A is non-singular and hence A^{-1} exist.

Now adj. A = $\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$ And $A^{-1} = \frac{1}{|A|}\text{adj.}A = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

6. Let A = $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ $\therefore$ $|A| = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} = -2-(-15) = -2+15 = 13 \neq 0$

$\therefore$ Matrix A is non-singular and hence A^{-1} exist.

Now adj. A = $\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$ And $A^{-1} = \frac{1}{|A|}\text{adj.}A = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

7. Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix} = 1(10-0)-2(0-0)+3(-0) = 10 \neq 0$

$\therefore$ A^{-1} exists.

$A_{11} = +\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = +(10-0) = 10,$ $A_{12} = -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = -(0-0) = 0,$

$A_{13} = +\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = +(0-0) = 0,$ $A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -(10-0) = -10,$

$A_{22} = +\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = +(5-0) = 5,$ $A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -(0-0) = 0,$

$A_{31} = +\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = +(8-6) = 2$, $\qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -(4-0) = -4$,

$A_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = +(2-0) = 2$

$\therefore \quad \text{adj. A} = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj.A} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

8. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ $\qquad \therefore \quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{vmatrix} = 1(-3-0)-0+0 = -3 \neq 0$

$\therefore \quad A^{-1}$ exists.

$A_{11} = +\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = +(-3-0) = -3$, $\qquad A_{12} = -\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = -(-3-0) = 3$,

$A_{13} = +\begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = +(6-15) = -9$, $\qquad A_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = -(0-0) = 0$,

$A_{22} = +\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = +(-1-0) = -1$, $\qquad A_{23} = -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -(2-0) = -2$,

$A_{31} = +\begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = +(0-0) = 0$, $\qquad A_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = -(0-0) = 0$,

$A_{33} = +\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = +(3-0) = 3$

$\therefore \quad \text{adj. A} = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj.A} = \frac{-1}{3} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$

9. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ $\qquad \therefore \quad |A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{vmatrix} = 2\{(-1)-(4)\}+3(8-7) = -3 \neq 0$

$\therefore$ A^{-1} exists.

$A_{11} = +\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = +(-1-0) = -1$, $A_{12} = -\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -(4-0) = -4$,

$A_{13} = +\begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = +(8-7) = 1$, $A_{21} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -(1-6) = 5$,

$A_{22} = +\begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = +(2+21) = 23$, $A_{23} = -\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -(4+7) = -11$,

$A_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = +(0+3) = 3$, $A_{32} = -\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -(0-12) = 12$,

$A_{33} = +\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = +(-2-4) = -6$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} -1 & 4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj.}A = \frac{-1}{3} = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix} = 1(8-6)-(-1)(0+9)+2(0-6) = -1 \neq 0$

$\therefore$ A^{-1} exists.

$A_{11} = +\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = +(8-6) = 2$, $A_{12} = -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9$,

$A_{13} = +\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = +(0-6) = -6$, $A_{21} = -\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = -(-4+4) = 0$,

$A_{22} = +\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = +(4-6) = -2$, $A_{23} = -\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -(-2+3) = -1$,

$A_{31} = +\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = +(3-4) = -1$, $A_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3$,

$A_{33} = +\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = +(2-0) = 2$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}.A = \frac{-1}{1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

11. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$ $\quad \therefore \quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix}$

$= 1\left(-\cos^2\alpha - \sin^2\alpha\right) - 0 + 0 = -\left(\cos^2\alpha + \sin^2\alpha\right) = -1 \neq 0$

$\therefore \quad A^{-1}$ exists.

$A_{11} = +\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix} = +\left(-\cos^2\alpha - \sin^2\alpha\right) = -\left(\cos^2\alpha + \sin^2\alpha\right) = -1,$

$A_{12} = -\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = -(0-0) = 0,$ $\quad A_{13} = +\begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = +(0-0) = 0,$

$A_{21} = -\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -(0-0) = 0,$ $\quad A_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = +(-\cos\alpha - 0) = -\cos\alpha,$

$A_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -(\sin\alpha - 0) = \sin\alpha,$ $\quad A_{31} = +\begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = (0-0) = 0,$

$A_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -(\sin\alpha - 0) = -\sin\alpha,$ $\quad A_{33} = +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = +(\cos\alpha - 0) = \cos\alpha$

$\therefore \quad \text{adj. } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}.A = -\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

12. Given: Matrix $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ $\quad \therefore \quad |A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix}$ = 15 – 14 = 1 $\neq$ 0

$\therefore \quad A^{-1} = \frac{1}{|A|}$ adj. A = $\frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Matrix $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ $\quad \therefore \quad |B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix}$ = 54 – 56 = –2 $\neq$ 0

$\therefore \quad B^{-1} = \frac{1}{|B|}$ adj. B = $\frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

Now $AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$\therefore$ $|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 67(61) - 87(47) = 4087 - 4089 = -2 \neq 0$

Now L.H.S. = $(AB)^{-1} = \frac{1}{|AB|}$ adj. $(AB) = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$ ....(i)

R.H.S. = $B^{-1}A^{-1} = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$ ....(ii)

$\therefore$ From eq. (i) and (ii), we get L.H.S. = R.H.S. $\Rightarrow$ $(AB)^{-1} = B^{-1}A^{-1}$

13. Given: $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ $\therefore$ $A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

L.H.S. = $A^2 - 5A + 7I = A^2 - 5A + 7I_2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8-15 & 5-5 \\ -5+5 & 3-10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} -7+7 & 0+0 \\ 0+0 & -7+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ = R.H.S. $\Rightarrow$ $A^2 - 5A + 7I_2 = 0$ ......(i)

To find: A^{-1}, multiplying eq. (i) by A^{-1}.

$\Rightarrow$ $A^2A^{-1} - 5A.A^{-1} + 7I_2A^{-1} = 0.A^{-1}$ $\Rightarrow$ $A - 5I_2 + 7A^{-1} = 0$

$\Rightarrow$ $7A^{-1} = -A + 5I_2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow$ $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

14. Given: $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\therefore$ $A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

$\therefore$ $A^2 + aA + bI_2 = 0$ $\Rightarrow$ $\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$$\Rightarrow \quad \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 11+3a+b & 8+2a+0 \\ 4+a+0 & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ We have $11+3a+b=0$ ....(i)

$8+2a+0=0 \quad \Rightarrow \quad 2a=-8 \quad \Rightarrow \quad a=-4$

Here $a=-4$ satisfies $4+a+0=0$ also, therefore $a=-4$

Putting $a=-4$ in eq. (i), $11-12+b=0 \Rightarrow b-1=0 \quad \Rightarrow \quad b=1$

Here also $b=1$ satisfies $3+a+b=0$, therefore $b=1$

Therefore, $a=-4$ and $b=1$

15. Given: $\text{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \quad \therefore \quad \text{A}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$\Rightarrow \quad \text{A}^2 = \begin{bmatrix} 1+1+1 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

Now $\text{A}^3 = \text{A}^2\text{A} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

L.H.S. = $\text{A}^3 - 6\text{A}^2 + 5\text{A} + 11\text{I}$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5 & 7-12+5 & 1-6+5 \\ -23+18+5 & 27-48+10 & -69+84-15 \\ 32-42+10 & -13+18-5 & 58-84+15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{R.H.S.}$$

Now, to find A^{-1}, multiplying $A^3 - 6A^2 + 5A + 11I = 0$ by A^{-1}

$\Rightarrow \quad A^3A^{-1} - 6A^2A^{-1} + 5AA^{-1} + 11I.A^{-1} = 0.A^{-1}$

$\Rightarrow \quad A^2 - 6A + 5I + 11A^{-1} = 0 \qquad \Rightarrow \quad 11A^{-1} = 6A - 5I - A^2$

$$\Rightarrow \quad 11A^{-1} = 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\Rightarrow \quad 11A^{-1} = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\Rightarrow \quad 11A^{-1} = \begin{bmatrix} 6-5-4 & 6-2 & 6-1 \\ 6+3 & 12-5-8 & -18+14 \\ 12-7 & -6+3 & 18-5-14 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

16. Given: $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \quad \therefore \quad A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 4+1+1 & -2-2-1 & 1+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Now $\quad A^3 = A^2A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

L.H.S. = $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36 & -21+30 & 21-30 \\ -21+30 & 22-36 & -21+30 \\ 21-30 & -21+30 & 22-36 \end{bmatrix} + \begin{bmatrix} 18-4 & -9-0 & 9-0 \\ -9-0 & 18-4 & -9-0 \\ 9-0 & -9-0 & 18-4 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix} + \begin{bmatrix} -14 & 9 & -9 \\ 9 & -14 & 9 \\ -9 & 9 & -14 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \text{R.H.S.}$$

Now, to find A^{-1}, multiplying $A^3 - 6A^2 + 9A - 4I$ by A^{-1}

$\Rightarrow \quad A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4I.A^{-1} = 0.A^{-1}$

$\Rightarrow \quad A^2 - 6A + 9I - 4A^{-1} = 0 \qquad \Rightarrow \quad 4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow \quad 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \quad 4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

17. If A is a non-singular matrix of order $n \times n$, then $|\text{adj. } A| = |A|^{n-1}$

$\therefore$ Putting $n = 3$, $|\text{adj. } A| = |A|^2$

Therefore, option (B) is correct.

18. Since $AA^{-1} = I$

$\therefore \quad |AA^{-1}| = |I| \quad \Rightarrow \quad |A||A^{-1}| = 1 \quad \Rightarrow \quad |A^{-1}| = \dfrac{1}{|A|}$

Therefore, option (B) is correct.

Exercise 4.6

Answers

1. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\therefore$ A = $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

Therefore, Unique solution and hence equations are consistent.

2. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\therefore$ A = $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2 - (-1) = 3 \neq 0$

Therefore, Unique solution and hence equations are consistent.

3. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$\therefore$ A = $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$ = 6 – 6 = 0

Now (adj. A) B = $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30-24 \\ -10+8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$

Therefore, given equations are inconsistent, i.e., have no common solution.

4. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Here A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{vmatrix}$

$\Rightarrow \quad |A| = 1(6a-2a)-1(4a-2a)+1(2a-3a) = 4a-2a-a = a \neq 0$

Therefore, Unique solution and hence equations are consistent.

5. Matrix form of given equations is AX = B $\Rightarrow \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Here $A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

$\Rightarrow \quad |A| = 3(0-5)-(-1)(0+3)+(-2)(0-6) = 3(-5)+3+12 = -15+15 = 0$

Now (adj. A) = $\begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$

And (adj. A) B = $\begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq 0$

Therefore, given equations are inconsistent.

6. Matrix form of given equations is AX = B $\Rightarrow \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

Here $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$

$\Rightarrow \quad |A| = 5(18+10)-(-1)(12-25)+4(-4-15) = 140-13-76 = 140-89 = 51 \neq 0$

Therefore, Unique solution and hence equations are consistent.

7. Matrix form of given equations is AX = B $\Rightarrow \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Here $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 5 & 2 \\ 7 & 5 \end{vmatrix} = 15-14 = 1 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, $x = 2$ and $y = -3$

8. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Here A = $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 8-(-3) = 8+3 = 11 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -5/11 \\ 12/11 \end{bmatrix}$$

Therefore, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$

9. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Here A = $\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 4 & -3 \\ 3 & -5 \end{vmatrix} = 8-(-3) = -20-(-9) = -20+9 = -11 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} -15+21 \\ -9+28 \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} 6 \\ 19 \end{bmatrix} = \begin{bmatrix} -6/11 \\ -19/11 \end{bmatrix}$$

Therefore, $x = \frac{-6}{11}$ and $y = \frac{-19}{11}$

10. Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Here A = $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\therefore \qquad |A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Therefore, $\qquad x = -1$ and $y = 4$

11. Matrix form of given equations is AX = B $\quad \Rightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$

Here $\quad A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$

$\therefore \qquad |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix} = 2(10+3) - 1(-5-0) + 1(3-0) = 26 + 5 + 3 = 34 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

Therefore, $\qquad x = 1, y = \frac{1}{2}$ and $z = \frac{3}{2}$

12. Matrix form of given equations is AX = B $\quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Here $\quad A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$\therefore \qquad |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1+3) - (-1)(2+3) + 1(2-1) = 4 + 5 + 1 = 10 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, $x = 2, y = -1$ and $z = 1$

13. Matrix form of given equations is AX = B $\Rightarrow \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

Here $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4+1)-3(-2-3)+3(-1+6) = 10+15+15 = 40 \neq 0$$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Therefore, $x = 1, y = 2$ and $z = -1$

14. Matrix form of given equations is AX = B $\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

Here $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix} = 1(12-5)-(-1)(9+10)+2(-3-8) = 7+9-22 = 4 \neq 0$$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Therefore, $x = 2, y = 1$ and $z = 3$

15. Given: Matrix A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$

$\Rightarrow$ $|A| = 2(-4+4) - (-3)(-6+4) + 5(3-2) = 0 - 6 + 5 = -1 \neq 0$

$\therefore$ A^{-1} exists and $A^{-1} = \frac{1}{|A|}(\text{adj. } A)$ ....(i)

Now, $A_{11} = 0, A_{12} = 2, A_{13} = 1$ and $A_{21} = -1, A_{22} = -9, A_{23} = -5$ and $A_{31} = 2, A_{32} = 23, A_{33} = 13$

$\therefore$ adj. A = $\begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$

$\therefore$ From eq. (i), $A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

Now, Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Here A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Therefore, solution is unique and $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, $x = 1, y = 2$ and $z = 3$

16. Let ₹x, ₹y, ₹z per kg be the prices of onion, wheat and rice respectively.

$\therefore$ According to given data, we have three equations,

$4x + 3y + 2z = 60$

$2x + 4y + 6z = 90$

$6x + 2y + 3z = 70$

Matrix form of given equations is AX = B $\Rightarrow$ $\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$

Here $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} = 4(12-12) - 3(6-36) + 2(4-24) = 0 + 90 - 40 = 50 \neq 0$

Therefore, solution is unique and $X = A^{-1}B = \frac{1}{|A|}(\text{adj. } A)B$ ....(i)

Now, $A_{11} = 0, A_{12} = 30, A_{13} = -20$

$A_{21} = -5, A_{22} = 0, A_{23} = 10$

$A_{31} = 10, A_{32} = -20, A_{33} = 10$

$\therefore$ adj. $A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}' = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$

$\Rightarrow$ From eq. (i), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$

$$= \frac{1}{50}\begin{bmatrix} -450+700 \\ 1800-1400 \\ -1200+900+700 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

Therefore, $x = 5, y = 8$ and $z = 8$

Hence, the cost of onion, wheat and rice are ₹ 5, ₹ 8 and ₹ 8 per kg.

Answers

1. Let $\Delta = \begin{bmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{bmatrix}$

Expanding along first row,

$$\Delta = x\begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta\begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta\begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix}$$

$\Rightarrow \quad \Delta = x\left(-x^2 - 1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$

$\Rightarrow \quad \Delta = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$\Rightarrow \quad \Delta = -x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right) = -x^3 - x + x = -x^3$ which is independent of θ.

2. L.H.S. = $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

Multiplying R_1 by a, R_2 by b and R_3 by c, $\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} = \dfrac{abc}{abc}\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$

$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix}$ [Interchanging C_1 and C_3]

$= (-)(-)\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$ [Interchanging C_2 and C_3] Proved.

3. Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along first row,

$= \cos\alpha\cos\beta\left(\cos\alpha\cos\beta - 0\right) - \cos\alpha\sin\beta\left(-\cos\alpha\sin\beta - 0\right) - \sin\alpha\left(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta\right)$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right)$

$= \cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)+\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)$

$= \cos^2\alpha+\sin^2\alpha$

= 1

4. Given: $\Delta=\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0$

$\Rightarrow \quad \Delta=\begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0 \quad [R_1 \to R_1+R_2+R_3]$

$\Rightarrow \quad \Delta=2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0$

Here, Either $2(a+b+c)=0 \quad \Rightarrow \quad a+b+c=0$ ……….(i)

Or $\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}=0$

$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ c+a & a+b-c-a & b+c-c-a \\ a+b & b+c-a-b & c+a-a-b \end{vmatrix}=0 \quad [C_2 \to C_2-C_1 \text{ and } C_3 \to C_3-C_1]$

$\Rightarrow \quad \begin{vmatrix} b-c & b-a \\ c-a & c-b \end{vmatrix}=0$ [Expanding along first row]

$\Rightarrow \quad (b-c)(c-b)-(b-a)(c-a)=0$

$\Rightarrow \quad bc-b^2-c^2+bc-bc+ab+ac-a^2=0$

$\Rightarrow \quad -a^2-b^2-c^2+ab+bc+ca=0$

$\Rightarrow \quad 2a^2+2b^2+2c^2-2ab-2bc-2ca=0$

$\Rightarrow \quad a^2+a^2+b^2+b^2+c^2+c^2-2ab-2bc-2ca=0$

$\Rightarrow \quad \left(a^2+b^2-2ab\right)+\left(b^2+c^2-2bc\right)+\left(a^2+c^2-2ca\right)=0$

$\Rightarrow \quad (a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow \quad a-b=0$ and $b-c=0$ and $c-a=0 \quad \left[x^2+y^2+z^2=0, \text{then } x=0, y=0, z=0\right]$

$\Rightarrow \quad a=b$ and $b=c$ and $c=a$ ……….(ii)

Therefore, from eq. (i) and (ii), either $a+b+c=0$ or $a=b=c$

5. Given: $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$ $\quad [R_1 \to R_1 + R_2 + R_3]$

$\Rightarrow \quad (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

Either $\quad 3x + a = 0 \quad \Rightarrow \quad x = \dfrac{-a}{3}$ ....(i)

Or $\quad \begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0 \qquad [C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$

$\Rightarrow \quad 1(a^2 - 0) = 0 \qquad \Rightarrow \quad a^2 = 0 \qquad \Rightarrow \quad a = 0$

But this is contrary as given that $a \neq 0$.

Therefore, from eq. (i), $x = \dfrac{-a}{3}$ is only the solution.

6. L.H.S. = $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$ = $\begin{vmatrix} a^2 & bc & c(a+c) \\ a(a+b) & b^2 & ac \\ ab & b(b+c) & c^2 \end{vmatrix}$

= $abc\begin{vmatrix} a & c & (a+c) \\ (a+b) & b & a \\ b & (b+c) & c \end{vmatrix}$

= $abc\begin{vmatrix} a-a-b-b & c-b-b-c & a+c-a-c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ $\qquad [R_1 \to R_1 - R_2 - R_3]$

= $abc\begin{vmatrix} -2b & -2b & 0 \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$ = $abc\begin{vmatrix} -2b & 0 & 0 \\ a+b & -a & a \\ b & c & c \end{vmatrix}$ $\qquad [C_2 \to C_2 - C_1]$

= $abc(-2b)(-ac-ac)$ = $4a^2b^2c^2$ = R.H.S. Proved.

7. Given: $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Since, $(AB)^{-1} = B^{-1}A^{-1}$ [Reversal law] ....(i)

Now $|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1(3-0)-2(-1-0)+(-2)(2-0) = 3+2-4=1\neq 0$

Therefore, B^{-1} exists.

$\therefore$ $B_{11}=3, B_{12}=1, B_{13}=2$ and $B_{21}=2, B_{22}=1, B_{23}=2$ and $B_{31}=6, B_{32}=2, B_{33}=5$

$\therefore$ adj. B = $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

$\therefore$ $B^{-1} = \frac{1}{|B|}(\text{adj. B}) = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

From eq. (i), $(AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$\Rightarrow$ $(AB)^{-1} = \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

8. Given: Matrix A = $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ $\therefore$ $|A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$

$\Rightarrow$ $|A| = 1(15-1)-(-2)(-10-1)+1(-2-3) = 14-22-5=-13\neq 0$

Therefore, A^{-1} exists.

$\therefore$ $A_{11}=14, A_{12}=11, A_{13}=-5$ and $A_{21}=11, A_{22}=4, A_{23}=-3$

and $A_{31}=-5, A_{32}=-3, A_{33}=-1$

$\therefore$ adj. A = $\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$ = B (say)

$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj. } A) = \frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$ ....(i)

$\Rightarrow \quad |B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix} = 14(-4-9)-11(-11-15)-5(-33+20) = 169 \neq 0$

Therefore, B^{-1} exists.

$\therefore \quad B_{11} = -13, B_{12} = 26, B_{13} = -13$ and $B_{21} = 26, B_{22} = -39, B_{23} = -13$

and $B_{31} = -13, B_{32} = -13, B_{33} = -65$

$\therefore \quad \text{adj. } B = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix} = -13\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$\Rightarrow B^{-1} = (\text{adj. } A)^{-1} = \frac{1}{|B|}(\text{adj. } B) = \frac{1}{169}(-13)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ ....(ii)

Now to find adj. A^{-1} = adj. C (say), where

$C = A^{-1} = \frac{-1}{13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -14/13 & -11/13 & 5/13 \\ -11/13 & -4/11 & 3/13 \\ 5/13 & 3/11 & 1/13 \end{bmatrix}$

$C = A^{-1} = \frac{-14}{13}\left(\frac{-4}{169} - \frac{9}{169}\right) - \left(\frac{-11}{13}\right)\left(\frac{-11}{169} - \frac{15}{169}\right) + \frac{5}{13}\left(\frac{-33}{169} + \frac{20}{169}\right)$

$C = A^{-1} = \frac{-14}{13}\left(\frac{-13}{169}\right) + \frac{11}{13}\left(\frac{-26}{169}\right) + \frac{5}{13}\left(\frac{-13}{169}\right) = \frac{14}{169} - \frac{22}{169} - \frac{5}{169} = \frac{-13}{169} = \frac{-1}{13} \neq 0$

Therefore, C^{-1} exists.

$\therefore \quad C_{11} = \frac{-1}{13}, C_{12} = \frac{2}{13}, C_{13} = \frac{-1}{13}$ and $C_{21} = \frac{2}{13}, C_{22} = \frac{-3}{13}, C_{23} = \frac{-1}{13}$

and $C_{31} = \frac{-1}{13}, C_{32} = \frac{-1}{13}, C_{33} = \frac{-5}{13}$

$\therefore \quad \text{adj. } A = \text{adj. } (A^{-1}) = \begin{vmatrix} -1/13 & 2/13 & -1/13 \\ 2/13 & -3/13 & -1/13 \\ -1/13 & -1/13 & -5/13 \end{vmatrix} = \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ ....(iii)

Again $\left(A^{-1}\right)^{-1} = C^{-1} = \frac{1}{|C|}(\text{adj. } C) = \frac{1}{-1/13}\left(\frac{-1}{13}\right)\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ = A (given)

(i) $(\text{adj. } A)^{-1} = \text{adj. } \left(A^{-1}\right)$

$$\Rightarrow \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$ [From eq. (ii) and (iii)]

(ii) $\left(A^{-1}\right)^{-1} = A$

$$\Rightarrow \frac{-1}{13}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

9. Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ $[R_1 \to R_1 + R_2 + R_3]$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x+y-y & x-y \\ x+y & x-x-y & y-x-y \end{vmatrix}$$ $[C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$

$$= 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix} = 2(x+y).1\begin{vmatrix} x & x-y \\ -y & -x \end{vmatrix}$$

$$= 2(x+y)\left\{-x^2 + y(x-y)\right\} = 2(x+y)\left(-x^2 + xy - y^2\right)$$

$$= -2(x+y)\left(x^2 - xy + y^2\right) = -2\left(x^3 + y^3\right)$$

10. Let $\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

$$= \begin{vmatrix} 1 & x & y \\ 0 & x+y-x & 0 \\ 0 & 0 & x+y-y \end{vmatrix}$$ $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

$$= \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = 1\begin{vmatrix} y & 0 \\ 0 & x \end{vmatrix} = xy$$

11. L.H.S. = $\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = \begin{vmatrix} \alpha & \alpha^2 & \alpha+\beta+\gamma \\ \beta & \beta^2 & \alpha+\beta+\gamma \\ \gamma & \gamma^2 & \alpha+\beta+\gamma \end{vmatrix}$ $[C_3 \to C_3 + C_1]$

$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta & \beta^2 & 1 \\ \gamma & \gamma^2 & 1 \end{vmatrix} = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \alpha^2 & 1 \\ \beta-\alpha & \beta^2-\alpha^2 & 0 \\ \gamma-\alpha & \gamma^2-\alpha^2 & 0 \end{vmatrix}$ $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

Expanding along third column, $(\alpha+\beta+\gamma)\begin{vmatrix} \beta-\alpha & \beta^2-\alpha^2 \\ \gamma-\alpha & \gamma^2-\alpha^2 \end{vmatrix}$

$= (\alpha+\beta+\gamma)\begin{vmatrix} \beta-\alpha & (\beta-\alpha)(\beta+\alpha) \\ \gamma-\alpha & (\gamma-\alpha)(\gamma+\alpha) \end{vmatrix} = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} 1 & (\beta+\alpha) \\ 1 & (\gamma+\alpha) \end{vmatrix}$

$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha) = (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)$

$= (\alpha+\beta+\gamma)[-(\alpha-\beta)](\gamma-\alpha)[-(\beta-\gamma)] = (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ = R.H.S.

12. L.H.S. = $\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} = \Delta_1 + \Delta_2$ (say) ....(i)

Now $\Delta_2 = \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} = pxyz\Delta_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -pxyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = pxyz\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$

$\Rightarrow$ From eq. (i), L.H.S. = $\Delta_1 + pxyz\Delta_1$ ....(ii)

Now $\Delta_1 = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 \\ y-x & y^2-x^2 & 0 \\ z-x & z^2-x^2 & 1 \end{vmatrix}$ $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

Expanding along third column, $\Delta_1 = \begin{vmatrix} y-x & y^2-x^2 \\ z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix}$

$= (y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} = (y-x)(z-x)(z+x-y-x) = (y-x)(z-x)(z-y)$

$= (x-y)(y-z)(z-x)$

$\therefore$ From eq. (i), L.H.S. = $(y-x)(z-x)(z-y)+pxyz(y-x)(z-x)(z-y)$

= $(1+pxyz)(y-x)(z-x)(z-y)$ = R.H.S.

13. L.H.S. = $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$ $[C_1 \to C_1+C_2+C_3]$

= $(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$

= $(a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 1 & -c+a & 2c+a \end{vmatrix}$ $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$

= $(a+b+c).1\begin{vmatrix} 2b+a & a-b \\ a-c & 2c+a \end{vmatrix}$ = $(a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$

= $(a+b+c)\left[4bc+2ab+a^2-a^2+ac+ab-bc\right]$ = $(a+b+c)(3ab+3bc+3ac)$

= $3(a+b+c)(ab+bc+ac)$ = R.H.S.

14. L.H.S. = $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

= $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$ $[R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 3R_1]$

= $1\begin{vmatrix} 1 & 2+p \\ 3 & 7+3p \end{vmatrix} - 0 + 0$ = $7+3p-3(2+p)$ = $7+3p-6-3p$ = 1 = R.H.S.

15. L.H.S. = $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta \end{vmatrix}$

= $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta-\sin\alpha\sin\delta+\sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta-\sin\beta\sin\delta+\sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta-\sin\gamma\sin\delta+\sin\gamma\sin\delta \end{vmatrix}$ $\left[C_3 \to C_3 + (\sin\delta)C_1\right]$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta \end{vmatrix} = \cos\delta \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha \\ \sin\beta & \cos\beta & \cos\beta \\ \sin\gamma & \cos\gamma & \cos\gamma \end{vmatrix}$$

$= \cos\delta(0)$ [$\because$ C_2 and C_3 have become identical]

= 0 = R.H.S.

16. Putting $\frac{1}{x} = u, \frac{1}{y} = v$ and $\frac{1}{z} = w$ in the given equations,

$2u + 3v + 10w = 4;\qquad 4u - 6v + 5w = 1;\qquad 6u + 9v - 20w = 2$

$\therefore$ the matrix form of given equations is $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ [AX= B]

Here, A = $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, X = $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36) = 150 + 330 + 750 = 1200 \neq 0$

$\therefore$ A^{-1} exists and unique solution is $X = A^{-1}B$ ....(i)

Now $A_{11} = 75, A_{12} = 110, A_{13} = 72$ and $A_{21} = 150, A_{22} = -100, A_{23} = 0$

and $A_{31} = 75, A_{32} = 30, A_{33} = -24$

$\therefore$ adj. A = $\begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

And $A^{-1} = \frac{\text{adj.}A}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\therefore$ From eq. (i), $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$

$\therefore$ $u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{5} \quad\Rightarrow\quad x = \frac{1}{u} = 2, y = \frac{1}{v} = 3, z = \frac{1}{w} = 5$

17. According to question, $b-a=c-b$ ....(i)

Let $\Delta=\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ = $\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(c-b) \end{vmatrix}$ $[R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_2]$

= $\begin{vmatrix} x+2 & x+3 & x+2a \\ 1 & 1 & 2(b-a) \\ 1 & 1 & 2(b-a) \end{vmatrix}$ [From eq. (i)] = 0 [$\because$ R_2 and R_3 have become identical]

Therefore, option (A) is correct.

18. Given: Matrix A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ $\therefore$ $|A|=\begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$

$\Rightarrow$ $|A| = x(yz-0)-0+0 = xyz \neq 0$

$\therefore$ A^{-1} exists and unique solution is $X = A^{-1}B$ ....(i)

Now $A_{11}=yz, A_{12}=0, A_{13}=0$ and $A_{21}=0, A_{22}=xz, A_{23}=0$ and $A_{31}=0, A_{32}=0, A_{33}=xy$

$\therefore$ adj. A = $\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}'$ = $\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$

And $A^{-1}=\dfrac{\text{adj.A}}{|A|}=\dfrac{1}{xyz}\begin{vmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{vmatrix}$ = $\begin{vmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{vmatrix}$ = $\begin{vmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{vmatrix}$ = $\begin{vmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{vmatrix}$

Therefore, option (A) is correct.

19. Given: Matrix A = $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$ $\therefore$ $|A|=\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$\Rightarrow$ $|A| = 1(1+\sin^2\theta)-\sin\theta(-\sin\theta+\sin\theta)+1(\sin^2\theta+1)$

$\Rightarrow$ $|A| = 1+\sin^2\theta+1+\sin^2\theta = 2+2\sin^2\theta$ ....(i)

Since $-1\le\sin\theta\le1$ $\Rightarrow$ $0\le\sin^2\theta\le1$ [$\because$ $\sin^2\theta$ cannot be negative]

$\Rightarrow$ $0\le 2\sin^2\theta\le2$ $\Rightarrow$ $2\le 2+2\sin^2\theta\le4$ $\Rightarrow$ $2\le \text{Det. A}\le4$

Therefore, option (D) is correct.